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WHITE NOISE FUNCTIONAL SOLUTIONS FOR WICK-TYPE STOCHASTIC COUPLED KdV EQUATIONS

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Abstract

Wick-type stochastic coupled KdV equations are researched. By means of Hermite transformation, white noise theory and modified tanh-coth method, four types of exact solutions to the stochastic coupled KdV equations are explicitly given. These solutions include the functional solutions of exponential type, hyperbolic type, trigonometric type, and quadratic trigonometric type.

1. Introduction

In this paper, we shall explore exact solutions for the following variable coefficients coupled KdV equations:

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$$\begin{cases} u_t + p(t)uu_x + q(t)vv_x + r(t)u_{xxx} = 0, \\ v_t + v_{xxx} - 3uv_x = 0, \end{cases} \tag{1.1}$$

where $p(t)$, $q(t)$, and $r(t)$ are bounded measurable or integrable functions on $\mathbb{R}_+$. Random wave is an important subject of stochastic PDEs. Many authors have studied this subject. Wadati first introduced and studied the stochastic KdV equations and gave the diffusion of soliton of the KdV equation under Gaussian noise in [14] and others [5-8, 10, 12] also researched stochastic KdV-type equations. Xie first introduced Wick-type stochastic KdV equations on white noise space and showed the auto-Bäcklund transformation and the exact white noise functional solutions in [17]. Furthermore, Chen and Xie [2-4] and Xie [18-23] researched some Wick-type stochastic wave equations by using white noise analysis method. Recently, Uğurlu and Kaya [13] gave the tanh function method, Wazzan [16] showed the modified tanh-coth method, and these methods have been applied to derive nonlinear transformations and exact solutions of nonlinear PDEs in mathematical physics. Many authors considered nonlinear wave PDEs, say, in two variables

$$A(u, u_t, u_x, u_{xt}, u_{xx}, u_{xxx}, \dots) = 0, \tag{1.2}$$

where A is a nonlinear function with respect to the indicated variables. Equation (1.2) can be converted to an ODE

$$B(u, u', u'', u''', \dots) = 0, \tag{1.3}$$

upon using a wave variable $\xi = x - \mu t$. Equation (1.3) is then integrated as long as all terms contain derivatives, where integration constants are considered zeros. The resulting ODE is then solved by the modified tanh-coth method [16], which admits the use of a finite series of functions of the form

$$u(x, t) = u(\xi) = \sum_{k=0}^{M} a_k Y^k(\xi) + \sum_{k=1}^{M} b_k Y^{-k}(\xi), \tag{1.4}$$

and the Riccati equation

$$Y' = \alpha + \beta Y + \gamma Y^2, \tag{1.5}$$

where α, β, and γ are constants to be prescribed later. The parameter M is a positive constant that can be determined by balancing the linear term of highest order with the nonlinear term in (1.3). Inserting (1.4) into the ODE in (1.3) and using (1.5) will give an algebraic equation in powers of Y. Since all coefficients of Y^k must vanish, this will give a system of algebraic equations with respect to parameters a_k and b_k. With the aid of Mathematica, we can determine a_k and b_k. If Equation (1.1) is considered in a random environment, we can get stochastic coupled KdV equations. In order to give the exact solutions of stochastic coupled KdV equations, we only consider this problem in white noise environment. We shall study the following Wick-type stochastic coupled KdV equations:

$$\begin{cases} U_t + P(t) \diamond U \diamond U_x + Q(t) \diamond V \diamond V_x + R(t) \diamond U_{xxx} = 0, \\ V_t + V_{xxx} - 3U \diamond V_x = 0, \end{cases} \tag{1.6}$$

and give white noise functional solutions, where "$\diamond$" is the Wick product on the Kondratiev distribution space $(\mathcal{S})_{-1}$, which was defined in [9], $P(t)$, $Q(t)$, and $R(t)$ are $(\mathcal{S})_{-1}$-valued functions.

2. White Noise Functional Solutions of Equation (1.6)

Taking the Hermite transform of Equation (1.6), we get the deterministic equations

$$\begin{cases} \widetilde{U}_t(x, t, z) + \widetilde{P}(t, z)\widetilde{U}(x, t, z)\widetilde{U}_x(x, t, z) + \widetilde{Q}(t, z)\widetilde{V}(x, t, z)\widetilde{V}_x(x, t, z) + \widetilde{R}(t, z)\widetilde{U}_{xxx}(x, t, z) = 0, \\ \widetilde{V}_t(x, t, z) + \widetilde{V}_{xxx}(x, t, z) - 3\widetilde{U}(x, t, z)\widetilde{V}_x(x, t, z) = 0, \end{cases} \tag{2.1}$$

where $z = (z_1, z_2, \dots) \in (\mathbb{C}^{\mathbb{N}})_c$ is a vector parameter.

To look for the travelling wave solution of Equation (2.1), we make the transformations $u(x, t, z) := \widetilde{U}(x, t, z) = \varphi(\xi(x, t, z))$, $v(x, t, z) := \widetilde{V}(x, t, z) = \psi(\xi(x, t, z))$, with

$$\xi(x, t, z) = kx + s\int_0^t \ell(\tau, z)d\tau + c,$$

where k, s, and c are arbitrary constants, which satisfy $ks \neq 0$, $\ell(\tau, z)$ is nonzero function of the indicated variables to be determined. So, Equation (2.1) can be changing into the form

$$\begin{cases} s\ell(t, z)\varphi' + kp(t, z)\varphi\varphi' + kq(t, z)\psi\psi' + k^3 r(t, z)\varphi''' = 0, \\ s\ell(t, z)\psi' + k^3\psi''' - 3k\varphi\psi' = 0, \end{cases} \tag{2.2}$$

where $p(t, z) := \widetilde{P}(t, z)$, $q(t, z) := \widetilde{Q}(t, z)$, and $r(t, z) := \widetilde{R}(t, z)$. Considering homogeneous balance between φ''', $\varphi\varphi'$ and $\psi\psi'$, ψ''' and $\varphi\psi'$ in turn, gives $M = 2$, hence we set the tanh-coth assumption by

$$\begin{cases} u(x, t, z) = \varphi(\xi) = a_0(t, z) + a_1(t, z)Y(\xi) + a_2(t, z)Y^2(\xi) \\ \qquad + b_1(t, z)Y^{-1}(\xi) + b_2(t, z)Y^{-2}(\xi), \\ v(x, t, z) = \psi(\xi) = a_0(t, z) + a_1(t, z)Y(\xi) + a_2(t, z)Y^2(\xi) \\ \qquad + b_1(t, z)Y^{-1}(\xi) + b_2(t, z)Y^{-2}(\xi), \end{cases} \tag{2.3}$$

where $Y(\xi)$ satisfies the Riccati equation (1.5).

Substituting (2.3) into (2.2) and using (1.5), collecting the coefficients of Y, all coefficients of Y^k have to vanish, these yields a system of algebraic equations in a_k, $c_k (k = 0, 1, 2)$, b_k, $d_k (k = 1, 2)$, and ℓ of the form

$$\begin{cases} s\ell G_0^{\varphi} + kp\zeta_0^{\varphi} + kq\zeta_0^{\psi} + k^3rK_0^{\varphi} = 0, \\ s\ell G_1^{\varphi} + kp\zeta_1^{\varphi} + kq\zeta_1^{\psi} + k^3rK_1^{\varphi} = 0, \\ s\ell G_2^{\varphi} + kp\zeta_2^{\varphi} + kq\zeta_2^{\psi} + k^3rK_2^{\varphi} = 0, \\ s\ell G_3^{\varphi} + kp\zeta_3^{\varphi} + kq\zeta_3^{\psi} + k^3rK_3^{\varphi} = 0, \\ s\ell H_1^{\varphi} + kp\eta_1^{\varphi} + kq\eta_1^{\psi} + k^3rL_1^{\varphi} = 0, \\ s\ell H_2^{\varphi} + kp\eta_2^{\varphi} + kq\eta_2^{\psi} + k^3rL_2^{\varphi} = 0, \\ s\ell H_3^{\varphi} + kp\eta_3^{\varphi} + kq\eta_3^{\psi} + k^3rL_3^{\varphi} = 0, \\ kp\zeta_4^{\varphi} + kq\zeta_4^{\psi} + k^3rK_4^{\varphi} = 0, \\ kp\zeta_5^{\varphi} + kq\zeta_5^{\psi} + k^3rK_5^{\varphi} = 0, \\ kp\eta_4^{\varphi} + kq\eta_4^{\psi} + k^3rL_4^{\varphi} = 0, \\ kp\eta_5^{\varphi} + kq\eta_5^{\psi} + k^3rL_5^{\varphi} = 0, \end{cases} \tag{2.4}$$

and

$$\begin{cases} s\ell G_0^{\psi} + k^3K_0^{\psi} - 3k\rho_0 = 0, \\ s\ell G_1^{\psi} + k^3K_1^{\psi} - 3k\rho_1 = 0, \\ s\ell G_2^{\psi} + k^3K_2^{\psi} - 3k\rho_2 = 0, \\ s\ell G_3^{\psi} + k^3K_3^{\psi} - 3k\rho_3 = 0, \\ s\ell H_1^{\psi} + k^3L_1^{\psi} - 3k\lambda_1 = 0, \\ s\ell H_2^{\psi} + k^3L_2^{\psi} - 3k\lambda_2 = 0, \\ s\ell H_3^{\psi} + k^3L_3^{\psi} - 3k\lambda_3 = 0, \\ k^2K_4^{\psi} - 3\rho_4 = 0, \\ k^2K_5^{\psi} - 3\rho_5 = 0, \\ k^2L_4^{\psi} - 3\lambda_4 = 0, \\ k^2L_5^{\psi} - 3\lambda_5 = 0, \end{cases} \tag{2.5}$$

where

$$G_0^{\varphi} = \alpha a_1 - \gamma b_1,\; G_1^{\varphi} = 2\alpha a_2 + \beta a_1,\; G_2^{\varphi} = 2\beta a_2 + \gamma a_1,\; G_3^{\varphi} = 2\gamma a_2,$$

$$G_0^{\psi} = \alpha c_1 - \gamma d_1,\; G_1^{\psi} = 2\alpha c_2 + \beta c_1,\; G_2^{\psi} = 2\beta c_2 + \gamma c_1,\; G_3^{\psi} = 2\gamma c_2,$$

$$H_1^{\varphi} = -(2\gamma b_2 + \beta b_1),\; H_2^{\varphi} = -(2\beta b_2 + \alpha b_1),\; H_3^{\varphi} = -2\alpha b_2,$$

$$H_1^{\psi} = -(2\gamma d_2 + \beta d_1),\; H_2^{\psi} = -(2\beta d_2 + \alpha d_1),\; H_3^{\psi} = -2\alpha d_2,$$

$$K_0^{\varphi} = \alpha(2\alpha G_2^{\varphi} + \beta G_1^{\varphi}) + \gamma(2\gamma H_2^{\varphi} + \beta H_1^{\varphi}),$$

$$K_1^{\varphi} = 2\alpha(3\alpha G_3^{\varphi} + 2_2^{\varphi} + G_1^{\varphi}) + \beta(2\alpha G_2^{\varphi} + \beta G_1^{\varphi}),$$

$$K_2^{\varphi} = 3\alpha(3\beta G_3^{\varphi} + 2\gamma G_2^{\varphi}) + 2\beta(3\alpha G_3^{\varphi} + 2\beta G_2^{\varphi} + \gamma G_1^{\varphi}) + \gamma(2\alpha G_2^{\varphi} + \beta G_1^{\varphi}),$$

$$K_3^{\varphi} = 2\gamma(3\alpha G_3^{\varphi} + 2\beta G_2^{\varphi} + \gamma G_1^{\varphi}) + 3\beta(3\beta G_3^{\varphi} + 2\gamma G_2^{\varphi}) + 12\alpha\gamma G_3^{\varphi},$$

$$K_4^{\varphi} = 3\gamma(3\beta G_3^{\varphi} + 2\gamma G_2^{\varphi}) + 12\beta\gamma G_3^{\varphi},\; K_5^{\varphi} = 12\gamma^2 G_3^{\varphi},$$

$$K_0^{\psi} = \alpha(2\alpha G_2^{\psi} + \beta G_1^{\psi}) + \gamma(2\gamma H_2^{\psi} + \beta H_1^{\psi}),$$

$$K_1^{\psi} = 2\alpha(3\alpha G_3^{\psi} + 2\beta G_2^{\psi} + \gamma G_1^{\psi}) + \beta(2\alpha G_2^{\psi} + \beta G_1^{\psi}),$$

$$K_2^{\psi} = 3\alpha(3\beta G_3^{\psi} + 2\gamma G_2^{\psi}) + 2\beta(3\alpha G_3^{\psi} + 2\beta G_2^{\psi} + \gamma G_1^{\psi}) + \gamma(2_2^{\psi} + \beta G_1^{\psi}),$$

$$K_3^{\psi} = 2\gamma(3\alpha G_3^{\psi} + 2\beta G_2^{\psi} + \gamma G_1^{\psi}) + 3\beta(3\beta G_3^{\psi} + 2\gamma G_2^{\psi}) + 12\alpha_3^{\psi},$$

$$K_4^{\psi} = 3\gamma(3\beta G_3^{\psi} + 2\gamma G_2^{\psi}) + 12\beta\gamma G_3^{\psi},\; K_5^{\psi} = 12\gamma^2 G_3^{\psi},$$

$$L_1^{\varphi} = 2\gamma(3\gamma H_3^{\varphi} + 2\beta H_2^{\varphi} + \alpha H_1^{\varphi}) + \beta(2_2^{\varphi} + \beta H_1^{\varphi}),$$

$$L_2^{\varphi} = 3\gamma(3\beta H_3^{\varphi} + 2\alpha H_2^{\varphi}) + 2\beta(3\gamma H_3^{\varphi} + 2\beta H_2^{\varphi} + \alpha H_1^{\varphi}) + \alpha(2\gamma H_2^{\varphi} + \beta H_1^{\varphi}),$$

$$L_3^{\varphi} = 2\alpha(3\gamma H_3^{\varphi} + 2\beta H_2^{\varphi} + \alpha H_1^{\varphi}) + 3\beta(3\beta H_3^{\varphi} + 2\alpha H_2^{\varphi}) + 12\alpha\gamma H_3^{\varphi},$$

$L_4^{\varphi} = 3\alpha(3_3^{\varphi} + 2\alpha H_2^{\varphi}) + 12\beta\alpha H_3^{\varphi},\ L_5^{\varphi} = 12\alpha^2 H_3^{\varphi},$

$L_1^{\psi} = 2h_2(3\gamma H_3^{\psi} + 2\beta H_2^{\psi} + \alpha H_1^{\psi}) + \beta(2\gamma H_2^{\psi} + \beta H_1^{\psi}),$

$L_2^{\psi} = 3\gamma(3\beta H_3^{\psi} + 2\alpha H_2^{\psi}) + 2\beta(3\gamma H_3^{\psi} + 2\beta H_2^{\psi} + \alpha H_1^{\psi}) + \alpha(2\gamma H_2^{\psi} + H_1^{\psi}),$

$L_3^{\psi} = 2\alpha(3\gamma H_3^{\psi} + 2\beta H_2^{\psi} + \alpha H_1^{\psi}) + 3\beta(3\beta H_3^{\psi} + 2\alpha H_2^{\psi}) + 12\alpha\gamma H_3^{\psi},$

$L_4^{\psi} = 3\alpha(3\beta H_3^{\psi} + 2\alpha H_2^{\psi}) + 12\beta\alpha H_3^{\psi},$

$L_5^{\psi} = 12\alpha^2 H_3^{\psi},\ \zeta_0^{\varphi} = a_0 G_0^{\varphi} + a_1 H_1^{\varphi} + a_2 H_2^{\varphi} + b_1 G_1^{\varphi} + b_2 G_2^{\varphi},$

$\zeta_1^{\varphi} = a_0 G_1^{\varphi} + a_1 G_0^{\varphi} + a_2 H_1^{\varphi} + b_1 G_2^{\varphi} + b_2 G_3^{\varphi},$

$\zeta_2^{\varphi} = a_0 G_2^{\varphi} + a_1 G_1^{\varphi} + a_2 G_0^{\varphi} + b_1 G_3^{\varphi},\ \zeta_3^{\varphi} = a_0 G_3^{\varphi} + a_1 G_2^{\varphi} + a_2 G_1^{\varphi},$

$\zeta_4^{\varphi} = a_1 G_3^{\varphi} + a_2 G_2^{\varphi},\ \zeta_5^{\varphi} = a_2 G_3^{\varphi},$

$\eta_1^{\varphi} = a_0 H_1^{\varphi} + a_1 H_2^{\varphi} + a_2 H_3^{\varphi} + b_1 G_0^{\varphi} + b_2 G_1^{\varphi},$

$\eta_2^{\varphi} = a_0 H_2^{\varphi} + a_1 H_3^{\varphi} + b_2 G_0^{\varphi} + b_1 H_1^{\varphi},\ \eta_3^{\varphi} = a_0 H_3^{\varphi} + b_1 H_2^{\varphi} + b_2 H_1^{\varphi},$

$\eta_4^{\varphi} = b_1 H_3^{\varphi} + b_2 H_2^{\varphi},\ \eta_5^{\varphi} = b_2 H_3^{\varphi}, \zeta_0^{\psi} = c_0 G_0^{\psi} + c_1 H_1^{\psi} + c_2 H_2^{\psi} + d_1 G_1^{\psi} + d_2 G_2^{\psi},$

$\zeta_1^{\psi} = c_0 G_1^{\psi} + c_1 G_0^{\psi} + c_2 H_1^{\psi} + d_1 G_2^{\psi} + d_2 G_3^{\psi},\ \zeta_2^{\psi} = c_0 G_2^{\psi} + c_1 G_1^{\psi} + c_2 G_0^{\psi} + d_1 G_3^{\psi},$

$\zeta_3^{\psi} = c_0 G_3^{\psi} + c_1 G_2^{\psi} + c_2 G_1^{\psi},\ \zeta_4^{\psi} = c_1 G_3^{\psi} + c_2 G_2^{\psi},\ \zeta_5^{\psi} = c_2 G_3^{\psi},$

$\eta_1^{\psi} = c_0 H_1^{\psi} + c_1 H_2^{\psi} + c_2 H_3^{\psi} d_1 G_0^{\psi} + d_2 G_1^{\psi},\ \eta_2^{\psi} = c_0 H_2^{\psi} + c_1 H_3^{\psi} + d_2 G_0^{\psi} + d_1 H_1^{\psi},$

$\eta_3^{\psi} = c_0 H_3^{\psi} + d_1 H_2^{\psi} + d_2 H_1^{\psi},\ \eta_4^{\psi} = d_1 H_3^{\psi} + d_2 H_2^{\psi},\ \eta_5^{\psi} = d_2 H_3^{\psi},$

$\rho_0 = a_0 G_0^{\psi} + a_1 H_1^{\psi} + a_2 H_2^{\psi} + b_1 G_1^{\psi} + b_2 G_2^{\psi},$

$\rho_1 = a_0 G_1^{\psi} + a_1 G_0^{\psi} + a_2 H_1^{\psi} + b_1 G_2^{\psi} + b_2 G_3^{\psi},$

$\rho_2 = a_0 G_2^{\psi} + a_1 G_1^{\psi} + a_2 G_0^{\psi} + b_1 G_3^{\psi},\ \rho_3 = a_0 G_3^{\psi} + a_1 G_2^{\psi} + a_2 G_1^{\psi},$

$\rho_4 = a_1 G_1^{\psi} + a_2 G_2^{\psi},\ \rho_5 = a_2 G_3^{\psi},\ \lambda_1 = a_0 H_1^{\psi} + a_1 H_2^{\psi} + a_2 H_3^{\psi} + b_1 G_0^{\psi} + b_2 G_1^{\psi},$

$\lambda_2 = a_0 H_2^{\psi} + a_1 H_3^{\psi} + b_1 H_1^{\psi} + b_2 G_0^{\psi},\ \lambda_3 = a_0 H_3^{\psi} + b_1 H_2^{\psi} + b_2 H_1^{\psi},$

$\lambda_4 = b_1 H_3^{\psi} + b_2 H_2^{\psi},\ \lambda_5 = b_2 H_3^{\psi}.$

Case I. If we set $\alpha = \beta = 1,\ \gamma = 0$ in (1.5), and use Mathematica to solve the resulting system, we will obtain the following set of solutions:

$$a_0 = a_1 = a_2 = c_1 = c_2 = 0,$$

$$b_1 = \frac{k^4(10p + 30r - 3(p + 3r)) - \frac{\theta_1\theta_2}{\sqrt{q}}}{2k^2(p + 3r)},\quad b_2 = 4k^2,\ d_1 = \frac{2}{q}(-3\sqrt{q}\theta_1 + \theta_2),$$

$$d_2 = \frac{-4}{\sqrt{q}}\theta_1,\ c_0 = \frac{11\sqrt{q}\theta_1(5 + 11p - 7r - pr + 2r^2) + 5\theta_2(5 + p - 7r + pr)}{16(5p + p^2 + 15r + pr - 6r^2)},$$

and $\ell = \frac{-k^3p^2}{sq}$, where $\theta_1 = ik^2\sqrt{(p + 3r)},\ \theta_2 = i\sqrt{q(p + 3r)},\ q > 0$, and $p + 3r > 0$.

Substituting these values in (2.3) and $Y = e^{\xi} - 1$, we obtain functional solutions of exponential type

$$u_1(x, t, z) = \{\frac{7k^4[p(t, z) + 3r(t, z)] - \frac{\theta_1(t,z)\theta_2(t,z)}{\sqrt{q(t, z)}}}{2k^2[p(t, z) + 3r(t, z)]}[\exp[\xi_1(x, t, z)] - 1] + 4k^2\}$$

$$\{\exp[\xi_1(x, t, z)] - 1\}^{-2}, \tag{2.6}$$

$$v_1(x, t, z)$$

$$= \frac{[11\sqrt{q(t, z)}\theta_1(t, z) + 5\theta_2(t, z)][5 + p(t, z) - [7 + p(t, z) + 2r(t, z)]r(t, z)]}{16q(t, z)[5p(t, z) + p^2(t, z) + 15r(t, z) + p(t, z)r(t, z) - 6r^2(t, z)]}$$

$$+ \{\frac{2}{q(t, z)}[-3\sqrt{q(t, z)}\theta_1(t, z) + \theta_2(t, z)][\exp[\xi_1(x, t, z)] - 1]$$

$$-\frac{4}{\sqrt{q(t, z)}}\theta_1(t, z)\}\{\exp[\xi_1(x, t, z)] - 1\}^{-2}, \tag{2.7}$$

with

$$\xi_1(x, t, z) = kx - k^3\int_0^t \frac{p^2(\tau, z)}{q(\tau, z)}d\tau + c_1.$$

Case II. If we set $\alpha = \frac{1}{2}$, $\beta = 0$, and $\gamma = -\frac{1}{2}$, in (1.5), and use Mathematica to solve the resulting system, we will obtain the following set of solutions:

$$a_1 = a_2 = b_1 = c_1 = c_2 = d_1 = 0,\ a_0 = \frac{k^2r[13p - 3kp + 10kq]}{2p[p + kq]},$$

$$b_2 = -d_2 = \frac{-3k^3r}{p + kq},\ c_0 = \frac{3k^2r}{q},\ \text{and}\ \ell = \frac{3k^3(k-1)rp}{2s(p + kq)},\ \text{where}\ pq \neq 0.$$

Substituting these values in (2.3), $Y(\xi) = \coth(\xi) \pm \operatorname{csch}(\xi)$ and $Y(\xi) = \tanh(\xi) \pm i\operatorname{sech}(\xi)$, we obtain functional solutions of hyperbolic type

$$u_2(x, t, z) = \frac{k^2r(t, z)[(13 - 3k)p(t, z) + 10kq(t, z)]}{2p(t, z)[p(t, z) + kq(t, z)]} - \frac{3k^3r(t, z)}{p(t, z) + kq(t, z)}$$

$$\{\coth[\xi_2(x, t, z)] \pm \operatorname{csch}[\xi_2(x, t, z)]\}^{-2}, \tag{2.8}$$

$$u_3(x, t, z) = \frac{k^2r(t, z)[(13 - 3k)p(t, z) + 10kq(t, z)]}{2p(t, z)[p(t, z) + kq(t, z)]} - \frac{3k^3r(t, z)}{p(t, z) + kq(t, z)}$$

$$\{\tanh[\xi_2(x, t, z)] \pm i\operatorname{sech}[\xi_2(x, t, z)]\}^{-2}, \tag{2.9}$$

$$v_2(x, t, z) = \frac{3k^2r(t, z)}{q(t, z)} + \frac{3k^3r(t, z)}{p(t, z) + kq(t, z)}$$

$$\{\coth[\xi_2(x, t, z)] \pm \operatorname{csch}[\xi_2(x, t, z)]\}^{-2}, \tag{2.10}$$

$$v_3(x, t, z) = \frac{3k^2r(t, z)}{q(t, z)} + \frac{3k^3r(t, z)}{p(t, z) + kq(t, z)}$$

$$\{\tanh[\xi_2(x, t, z)] \pm i\,\text{sech}[\xi_2(x, t, z)]\}^{-2}, \tag{2.11}$$

where

$$\xi_2(x, t, z) = kx + \frac{3}{2}k^3(k-1)\int_0^t \frac{r(\tau, z)p(\tau, z)d\tau}{p(\tau, z) + kq(\tau, z)} + c_2.$$

Case III. If we set $\alpha = 1, \beta = 0,$ and $\gamma = 1,$ in (1.5), and use Mathematica to solve the resulting system, we will obtain the following set of solutions:

$$a_0 = a_2 = b_1 = b_2 = c_2 = d_1 = d_2 = 0,\ a_1 = \frac{iq}{\sqrt{pq}}\delta,\ c_0 = \frac{2ik^2(1-4k)}{\sqrt{pq}},$$

$c_1 = \delta,$ and $\ell = \frac{-2k^3 p}{sq}$, where δ is an arbitrary constant and $pq > 0.$

Substituting these values in (2.3), $Y(\xi) = \tan(\xi)$ and $Y(\xi) = -\cot(\xi),$ we obtain functional solutions of trigonometric type

$$u_4(x, t, z) = \frac{iq(t, z)\delta}{\sqrt{p(t, z)q(t, z)}}\tan[\xi_3(x, t, z)], \tag{2.12}$$

$$u_5(x, t, z) = -\frac{iq(t, z)\delta}{\sqrt{p(t, z)q(t, z)}}\cot[\xi_3(x, t, z)], \tag{2.13}$$

$$v_4(x, t, z) = \frac{2ik^2(1-4k)}{\sqrt{p(t, z)q(t, z)}} + \delta\tan[\xi_3(x, t, z)], \tag{2.14}$$

$$v_5(x, t, z) = \frac{2ik^2(1-4k)}{\sqrt{p(t, z)q(t, z)}} - \delta\cot[\xi_3(x, t, z)], \tag{2.15}$$

where

$$\xi_3(x, t, z) = kx - 2k^3\int_0^t \frac{p(\tau, z)}{q(\tau, z)}d\tau + c.$$

Case IV. If we set $\alpha = 1, \beta = 0,$ and $\gamma = 4,$ in (1.5), and use Mathematica to solve the resulting system, we will obtain the following set of solutions:

$$a_1 = b_1 = b_2 = c_1 = c_0 = d_1 = d_2 = 0,$$

$$a_0 = k^2[\frac{32}{3} + \frac{k}{3 + kp}(16r - \frac{4093}{384}p)],\ a_2 = 64k^2,$$

$$c_2 = ik^2\sqrt{\frac{6(683p + 2048r)}{q}},\ \text{and}\ \ell = \frac{k^3}{s(k+3)}[48r - \frac{4093}{128}p],$$

where $q \neq 0$ and $683p + 2048r / q > 0$.

Substituting these values in (2.3), $Y(\xi) = \frac{1}{2}\cot(2\xi)$ and $Y(\xi) = \frac{1}{2}\tan(2\xi)$, we obtain functional solutions of quadratic trigonometric type

$$u_6(x, t, z) = k^2\{\frac{32}{3} + \frac{k}{3 + kp(t, z)}[16r(t, z) - \frac{4093}{384}p(t, z)]\}$$

$$+ 16k^2\cot^2[2\xi_4(x, t, z)], \tag{2.16}$$

$$u_7(x, t, z) = k^2\{\frac{32}{3} + \frac{k}{3 + kp(t, z)}[16r(t, z) - \frac{4093}{384}p(t, z)]\}$$

$$+ 16k^2\tan^2[2\xi_4(x, t, z)], \tag{2.17}$$

$$v_6(x, t, z) = \frac{ik^2}{4}\sqrt{\frac{6[683p(t, z) + 2048r(t, z)]}{q(t, z)}}\cot^2[2\xi_4(x, t, z)], \tag{2.18}$$

$$v_7(x, t, z) = \frac{ik^2}{4}\sqrt{\frac{6[683p(t, z) + 2048r(t, z)]}{q(t, z)}}\tan^2[2\xi_4(x, t, z)], \tag{2.19}$$

where

$$\xi_4(x, t, z) = kx + \frac{k^3}{k+3}\int_0^t[48r(\tau, z) - \frac{4093}{128}p(\tau, z)]d\tau + c.$$

Lemma 2.1 [9]. *Suppose $u(x, t, z)$ is a solution (in the usual strong, pointwise sense) of Equation (2.1) for (x, t) in some bounded open set $G \subset \mathbb{R} \times \mathbb{R}_+$, and for all $z \in K_m(n)$ for some m and n. Moreover, suppose that $u(x, t, z)$ and all its partial derivatives, which are involved in Equation (2.1), are (uniformly) bounded for $(x, t, z) \in G \times K_m(n)$, continuous with respect to $(x, t) \in G$ for all $z \in K_m(n)$ and analytic with* *respect to* $z \in K_m(n)$ *for all* $(x, t) \in G$. *Then, there exists* $U(x, t) \in (\mathcal{S})_{-1}$ *such that* $u(x, t, z) = \widetilde{U}(x, t)(z)$ *for all* $(x, t, z) \in G \times K_m(n)$ *and* $U(x, t)$ *solves* (*in the strong sense in* $(\mathcal{S})_{-1}$) *Equation* (1.6) *in* $(\mathcal{S})_{-1}$.

From Lemma 2.1, we know that there exists $U(x, t) \in (\mathcal{S})_{-1}$ such that $u(x, t, z) = \widetilde{U}(x, t)(z)$ for all $(x, t, z) \in G \times K_m(n)$, where $U(x, t)$ is the inverse Hermite transformation of $u(x, t, z)$. Consequently, $U(x, t)$ solves Equation (1.6), therefore, the white noise functional solutions of (1.6) are as follows:

2.1. White noise functional solutions of exponential type

For $Q(t) > 0$ and $P(t) + 3R(t) > 0$, we have that the solutions of (1.6) are the following:

$$U_1(x, t) = \{[7k^4[P(t) + 3R(t)] - \Theta_1(t) \diamond \Theta_2(t) \diamond Q(t)^{\diamond(-\frac{1}{2})}] \diamond [\frac{1}{2k^2}[P(t) + 3R(t)]^{\diamond(-1)}]$$

$$\diamond [\exp^{\diamond}[\Xi_1(x, t)] - 1] + 4k^2\} \diamond \{\exp^{\diamond}[\Xi_1(x, t)] - 1\}^{-2}, \tag{2.20}$$

$$V_1(x, t) = \frac{1}{16}[11Q^{\diamond(\frac{1}{2})}(t) \diamond \Theta_1(t) + 5\Theta_2(t)] \diamond [5 + P(t) - [7 + P(t) + 2R(t) + 2R(t)]$$

$$\diamond R(t)] \diamond Q^{\diamond(-1)}(t) \diamond [5P(t) + P^{\diamond 2}(t) + 15R(t) + P(t) \diamond R(t)$$

$$- 6R^{\diamond 2}(t)]^{\diamond(-1)} + \{2Q^{\diamond(-1)}(t) \diamond [-3Q^{\diamond(\frac{1}{2})}(t) \diamond \Theta_1(t) + \Theta_2(t)]$$

$$\diamond [\exp^{\diamond}[\Xi_1(x, t) - 1] - 4Q^{\diamond(-\frac{1}{2})}(t) \diamond \Theta_1(t)\} \diamond \{\exp^{\diamond}[\Xi_1(x, t)] - 1\}^{-2}, \tag{2.21}$$

with $\Theta_1(t) = ik^2[P(t) + 3R(t)]^{\diamond(\frac{1}{2})}$, $\Theta_2(t) = ik^2[Q(t) \diamond (P(t) + 3R(t))]^{\diamond(\frac{1}{2})}$, and

$$\Xi_1(x, t, z) = kx - k^3\int_0^t P^{\diamond 2}(\tau, z) \diamond Q^{\diamond(-1)}(\tau, z)d\tau + c_1.$$

2.2. White noise functional solutions of hyperbolic type

For $P(t) \diamond Q(t) \neq 0$, we have that the solutions of (1.6) are the following:

$$U_2(x, t) = \frac{1}{2} k^2 R(t) \diamond [(13 - 3k)P(t) + 10kQ(t)] \diamond [P^{\diamond 2}(t) + kP(t) \diamond Q(t)]^{\diamond(-1)}$$

$$- 3k^3 R(t) \diamond [P(t) + kQ(t)]^{\diamond(-1)}$$

$$\diamond \{ \coth^{\diamond}[\Xi_2(x, t)] \pm \operatorname{csch}^{\diamond}[\Xi_2(x, t)]\}^{\diamond(-2)}, \tag{2.22}$$

$$U_3(x, t) = \frac{1}{2} k^2 R(t) \diamond [(13 - 3k)P(t) + 10kQ(t)] \diamond [P^{\diamond 2}(t) + kP(t) \diamond Q(t)]^{\diamond(-1)}$$

$$- 3k^3 R(t) \diamond [P(t) + kQ(t)]^{\diamond(-1)}$$

$$\diamond \{ \tanh^{\diamond}[\Xi_2(x, t)] \pm i \operatorname{sech}^{\diamond}[\Xi_2(x, t)]\}^{\diamond(-2)}, \tag{2.23}$$

$$V_2(x, t) = 3k^2 R(t) \diamond Q(t)^{\diamond(-1)}(t) + 3k^3 R(t) \diamond [P(t) + kQ(t)]^{\diamond(-1)}$$

$$\diamond \{ \coth^{\diamond}[\Xi_2(x, t)] \pm \operatorname{csch}^{\diamond}[\Xi_2(x, t)]\}^{\diamond(-2)}, \tag{2.24}$$

$$V_3(x, t) = 3k^2 R(t) \diamond Q(t)^{\diamond(-1)}(t) + 3k^3 R(t) \diamond [P(t) + kQ(t)]^{\diamond(-1)}$$

$$\diamond \{ \tanh^{\diamond}[\Xi_2(x, t)] \pm i \operatorname{sech}^{\diamond}[\Xi_2(x, t)]\}^{\diamond(-2)}, \tag{2.25}$$

with

$$\Xi_2(x, t) = kx + \frac{3}{2} k^3 (k - 1) \int_0^t R(\tau) \diamond P(\tau) \diamond [P(\tau) + kQ(\tau)]^{\diamond(-1)} d\tau + c_2.$$

2.3. White noise functional solutions of trigonometric type

For $P(t) \diamond Q(t) > 0$, we have that the solutions of (1.6) are the following:

$$U_4(x, t) = \delta i Q(t) \diamond [P(t) \diamond Q(t)]^{\diamond(-\frac{1}{2})} \tan^{\diamond}[\Xi_3(x, t)], \tag{2.26}$$

$$U_5(x,\, t) = -\delta i Q(t) \diamond [P(t) \diamond Q(t)]^{\diamond(-\frac{1}{2})} \cot^{\diamond}[\Xi_3(x,\, t)], \tag{2.27}$$

$$V_4(x,\, t) = 2ik^2(1-4k)[P(t) \diamond Q(t)]^{\diamond(-\frac{1}{2})} + \delta \tan^{\diamond}[\Xi_3(x,\, t)], \tag{2.28}$$

$$V_5(x,\, t) = 2ik^2(1-4k)[P(t) \diamond Q(t)]^{\diamond(-\frac{1}{2})} - \delta \cot^{\diamond}[\Xi_3(x,\, t)], \tag{2.29}$$

with

$$\Xi_3(x,\, t,\, z) = kx - k^3 \int_0^t P(\tau,\, z) \diamond Q^{\diamond(-1)}(\tau,\, z) d\tau + c_3.$$

2.4. White noise functional solutions of quadratic trigonometric type

For $Q(t) \neq 0$ and $[683P(t) + 2048R(t)]Q^{\diamond(-1)}(t) > 0$, we have that the solutions of (1.6) are the following:

$$U_6(x,\, t) = k^2\{\frac{32}{3} + k[3 + kP(t)]^{\diamond(-1)} \diamond [16R(t) - \frac{4093}{384}P(t)]\} + 16k^2 \cot^{\diamond 2}[2\Xi_4(x,\, t)], \tag{2.30}$$

$$U_7(x,\, t) = k^2\{\frac{32}{3} + k[3 + kP(t)]^{\diamond(-1)} \diamond [16R(t) - \frac{4093}{384}P(t)]\} + 16k^2 \tan^{\diamond 2}[2\Xi_4(x,\, t)], \tag{2.31}$$

$$V_6(x,\, t) = \frac{\sqrt{6}ik^2}{4}[683P(t) + 2048R(t)]^{\diamond(\frac{1}{2})} \diamond Q(t)^{\diamond(-\frac{1}{2})} \diamond \cot^{\diamond 2}[2\Xi_4(x,\, t)], \tag{2.32}$$

$$V_7(x,\, t) = \frac{\sqrt{6}ik^2}{4}[683P(t) + 2048R(t)]^{\diamond(\frac{1}{2})} \diamond Q(t)^{\diamond(-\frac{1}{2})} \diamond \tan^{\diamond 2}[2\Xi_4(x,\, t)], \tag{2.33}$$

with

$$\Xi_4(x, t) = kx + \frac{k^3}{k+3}\int_0^t [48R(\tau) - \frac{4093}{128}P(\tau)]d\tau + c_4.$$

We observe that for different forms of $P(t)$, $Q(t)$, and $R(t)$, we can get different solutions of (1.6) from (2.20)-(2.33).

3. Remark

Let $W(t) = \dot{B}(t)$ be the Gaussian white noise, where $B(t)$ is the Brownian motion. We have the Hermite transform $\widetilde{W}(t, z) = \sum_{i=1}^{\infty} z_i \int_0^t \eta_i(s)ds$. Since $\exp^{\diamond}[B(t)] = \exp[B(t) - t^2/2]$, we have $\tan^{\diamond}[B(t)] = \tan[B(t) - t^2/2]$, $\cot^{\diamond}[B(t)] = \cot[B(t) - t^2/2]$, $\tanh^{\diamond}[B(t)] = \tanh[B(t) - t^2/2]$, $\coth^{\diamond}[B(t)] = \coth[B(t) - t^2/2]$, $\operatorname{sech}^{\diamond}[B(t)] = \operatorname{sech}[B(t) - t^2/2]$, and $\operatorname{csch}^{\diamond}[B(t)] = \operatorname{csch}[B(t) - t^2/2]$. Suppose $P(t) = R(t) = \sigma_1 Q(t)$ and $Q(t) = q(t) + \sigma_2 W(t)$, where σ_1 and σ_2 are arbitrary constants and $q(t)$ is integrable or bounded measurable function on $\mathbb{R}_+$. The white noise functional solutions of (1.6) are as follows:

$$U_8(x, t) = \frac{14k(1-k^2)(\exp[\Gamma_1(x, t)] - 1) + 4k^2}{(\exp[\Gamma_1(x, t)] - 1)^2}, \tag{3.1}$$

$$V_8(x, t) = 8ik^2\sqrt{\sigma_1}\{[20 - 36\sigma_1][q(t) + \sigma_2 W(t)] - 12\sigma_1^2[q(t) + \sigma_2 W(t)]^2 - \frac{\exp[\Gamma_1(x, t)]}{(\exp[\Gamma_1(x, t)] - 1)^2}\}, \tag{3.2}$$

with

$$\Gamma_1(x, t) = kx - k^3\sigma_1^2[\int_0^t q(\tau)d\tau + \sigma_2 B(t) - \frac{\sigma_2 t^2}{2}] + c_1,$$

$$U_9(x, t) = \frac{k^2[(13 - 3k)\sigma_1 + 10k]}{2(\sigma_1 + k)} - \frac{3k^2\sigma_1}{\sigma_1 + k}$$

$$\{ \coth[\Gamma_2(x, t)] \pm \operatorname{csch}[\Gamma_2(x, t)]\}, \tag{3.3}$$

$$U_{10}(x, t) = \frac{k^2[(13 - 3k)\sigma_1 + 10k]}{2(\sigma_1 + k)} - \frac{3k^2\sigma_1}{\sigma_1 + k}$$

$$\{ \tanh[\Gamma_2(x, t)] \pm i \operatorname{sech}[\Gamma_2(x, t)]\}, \tag{3.4}$$

$$V_9(x, t) = 3k^2\sigma_1 + \frac{3k^3\sigma_1}{\sigma_1 + k}\{ \coth[\Gamma_2(x, t)] \pm \operatorname{csch}[\Gamma_2(x, t)]\}, \tag{3.5}$$

$$V_{10}(x, t) = 3k^2\sigma_1 + \frac{3k^3\sigma_1}{\sigma_1 + k}\{ \tanh[\Gamma_2(x, t)] \pm i \operatorname{sech}[\Gamma_2(x, t)]\}, \tag{3.6}$$

with

$$\Gamma_2(x, t) = kx + \frac{3\sigma_1^2 k^3(k-1)}{2(\sigma_1 + k)}\left[\int_0^t q(\tau)d\tau + \sigma_2 B(t) - \frac{\sigma_2 t^2}{2}\right] + c_2,$$

$$U_{11}(x, t) = \frac{\delta i}{\sqrt{\sigma_1}} \tan[\Gamma_3(x, t)], \tag{3.7}$$

$$U_{12}(x, t) = -\frac{\delta i}{\sqrt{\sigma_1}} \cot[\Gamma_3(x, t)], \tag{3.8}$$

$$V_{11}(x, t) = \frac{2ik^2(1-4k)}{\sqrt{\sigma_1}[q(t) + \sigma_2 W(t)]} + \delta \tan[\Gamma_3(x, t)], \tag{3.9}$$

$$V_{12}(x, t) = \frac{2ik^2(1-4k)}{\sqrt{\sigma_1}[q(t) + \sigma_2 W(t)]} - \delta \cot[\Gamma_3(x, t)], \tag{3.10}$$

with

$$\Gamma_3(x, t) = kx - 2k^3\sigma_1 t + c_3,$$

$$U_{13}(x, t) = k^2\left\{\frac{32}{3} - \frac{2045\sigma_1 k[q(t) + \sigma_2 W(t)]}{3 + k\sigma_1[q(t) + \sigma_2 W(t)]}\right\} + 16k^2 \cot^2[2\Gamma_4(x, t)], \tag{3.11}$$

$$U_{14}(x,\, t) = k^2\left\{\frac{32}{3} - \frac{2045\sigma_1 k[q(t) + \sigma_2 W(t)]}{3 + k\sigma_1[q(t) + \sigma_2 W(t)]}\right\} + 16k^2 \tan^2[2\Gamma_4(x,\, t)],$$

(3.12)

$$V_{13}(x,\, t) = \frac{ik^2}{2}\sqrt{16386\sigma_1}\, \cot^2[2\Gamma_4(x,\, t)], \tag{3.13}$$

$$V_{14}(x,\, t) = \frac{ik^2}{2}\sqrt{16386\sigma_1}\, \tan^2[2\Gamma_4(x,\, t)], \tag{3.14}$$

with

$$\Gamma_4(x,\, t) = kx + \frac{2051\sigma_1 k^3}{128(k+3)}\left[\int_0^t q(\tau)d\tau + \sigma_2 B(t) - \frac{\sigma_2 t^2}{2}\right] + c_4.$$

4. Summary and Discussion

We have discussed the solutions of SPDEs driven by Gaussian white noise. There is a unitary mapping between the Gaussian white noise space and the Poisson white noise space. This connection was given by Benth and Gjerde [1]. We can see in the Subsection 4.9 [9] clearly. Hence, by the aid of the connection, we can derive some stochastic exact soliton solutions if the coefficients $P(t)$, $Q(t)$, and $R(t)$ are Poisson white noise functions in Equation (1.6). In this paper, using Hermite transformation, white noise theory, and modified tanh-coth method, we study the white noise solutions of the Wick-type stochastic coupled KdV equations. This paper shows that the modified tanh-coth method is sufficient to solve the stochastic nonlinear equations in mathematical physics. The method which we have proposed in this paper is standard, direct, and computerized method, which allows us to do complicated and tedious algebraic calculation. It is shown that the algorithm can be also applied to other NLPDEs in mathematical physics such as modified Hirota-Satsuma coupled KdV, KdV-Burgers, modified KdV-Burgers, Sawada-Kotera, Zhiber-Shabat equations, and Benjamin-Bona-Mahony equations. Since the Riccati equation has other solutions if select other values of α, β, and γ, there are many other exact solutions of variable coefficients and Wick-type stochastic coupled KdV equations.

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